Bang–bang and Singular Control Problems: Applications, Sufficient Conditions and Sensitivity Analysis

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Overview

Basic tasks for solving optimal control problems:

- Necessary conditions: Pontryagin's maximum principle
- Numerical methods: verify necessary conditions;
  (1) boundary value methods, (2) discretize and optimize
- (Second-order) Sufficient conditions
- Sensitivity analysis and real-time control techniques

Optimal control problems with control appearing linearly:

- bang–bang and singular control
- numerical methods: direct optimization of switching times
- Verification of second order sufficient conditions (SSC)
  - bang–bang control: Agrachev/Stefani/Zezza; Osmolovskii/Maurer
  - singular control: work in progress: Dmitruk, Stefani, Vossen
  - state constraints: Ledzewicz/Schättler, Maurer/Vossen
  - applications to sensitivity analysis

Overview

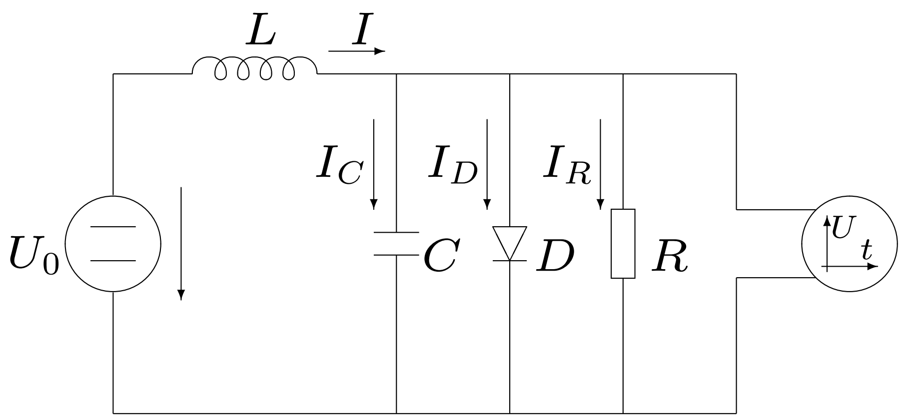
Examples

- Control of a Van der Pol oscillator: bang-bang and singular control
- Time–optimal control for a semiconductor laser
- Time –optimal control of a two-link robot
- An economic model for optimal production and maintenance
- GODDARD problem: bang-singular control

Joint work with

Nikolai Osmolovskii, Jang–Ho Robert Kim, Christof Büskens, Georg Vossen

Van der Pol oscillator: time-optimal control



$x_1(t) = U(t)$ voltage

$u(t) = U_0(t)$ control

Minimize the final time t_f

subject to

$$\begin{aligned}
&\dot{x}_1(t) = x_2(t),\\
&\dot{x}_2(t) = -x_1(t) + x_2(t)(\,p - x_1(t)^2) + u(t),\\
&x_1(0) = 1, \quad x_2(0) = 1, \quad x_1(t_f)^2 + x_2(t_f)^2 = r^2 \;\; (\,r = 0.2\,),\\
&-1 \le u(t) \le 1, \quad t \in [0, t_f].
\end{aligned}$$

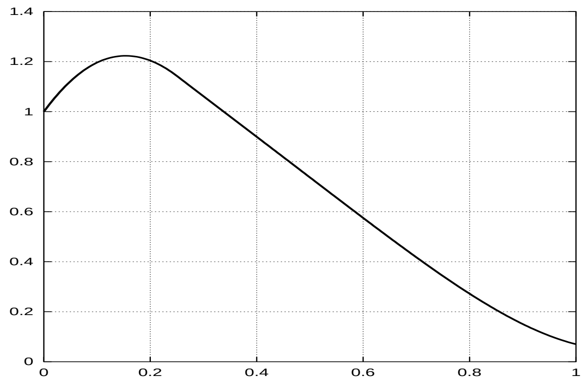
Perturbation p , nominal value $p_0 = 1.0$:

Optimal bang–bang control

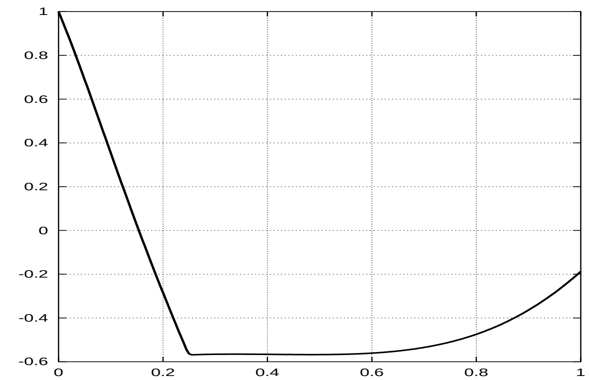
$$u(t) = \left\{ \begin{array}{rcll} -1 & , & \text{for} & 0 \le t \le t_1 \\ 1 & , & \text{for} & t_1 \le t \le t_f \end{array} \right\}$$

Nominal optimal solution for $p_0 = 1$

$x_1(t)$

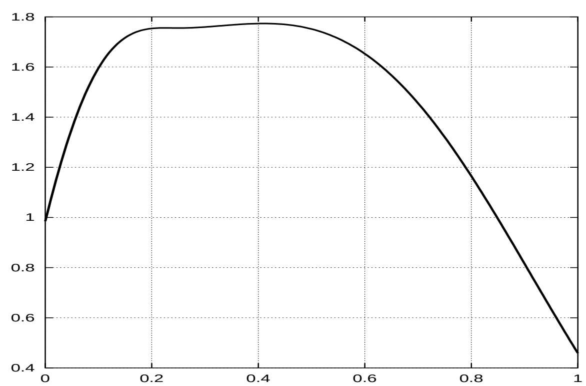


$x_2(t)$

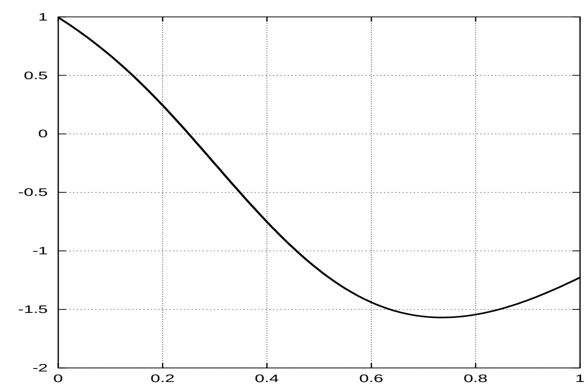


State trajectories $x_1(t),\ x_2(t)$

$\lambda_1(t)$



$\lambda_2(t)$



Adjoint variables $\lambda_1(t),\ \lambda_2(t) = \sigma(t)$ (switching function), $\dot{\sigma}(t_1) \neq 0$

SSC and sensitivity analysis

Optimization variables : $z := (t_1, t_f)$

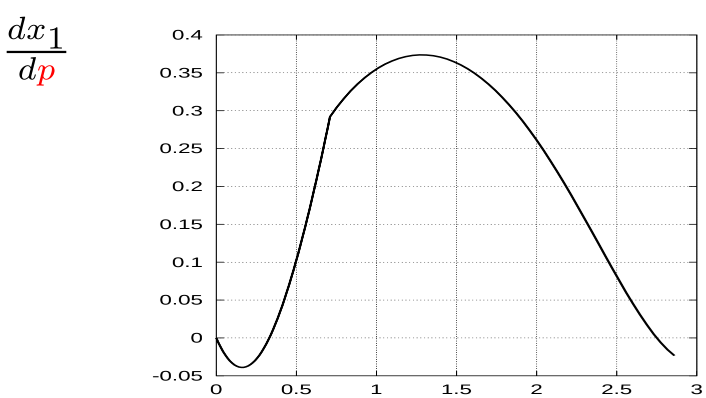
final time $t_f = 2.86419188$, switching time $t_1 = 0.713935566$.

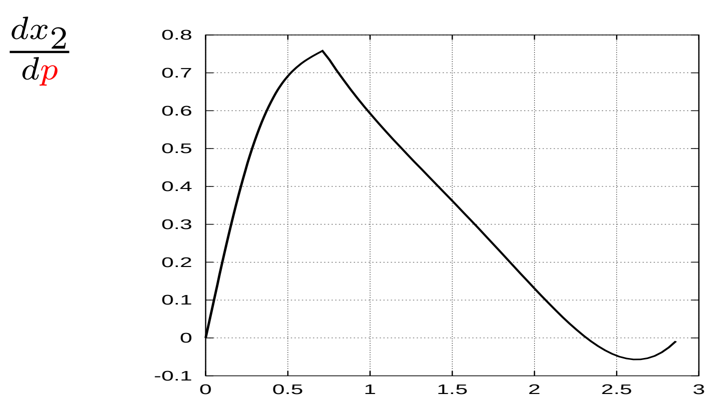
Compute Jacobian of terminal conditions and Hessian of Lagrangian:

$$\Phi_z = (-0.0000264, 0.3049115), \quad \mathcal{L}_{zz} = \begin{pmatrix} 188.066 & -7.39855 \\ -7.39855 & 3.06454 \end{pmatrix}$$

SSC hold ! Sensitivity derivatives exist (code NUDOCCCS, C. Büskens)

$$\frac{dt_1}{dp} = -0.344220, \qquad \frac{dt_f}{dp} = 1.395480$$





Parametric sensitivity derivatives of the state variables (scaled)

Van der Pol oscillator: singular control

Minimize	$J(x,u) = \frac{1}{2}\int_0^{t_f} (x_1^2 + x_2^2)dt\,,$	$t_f = 4,$
subject to	$\dot{x}_1 = x_2,$	$x_1(0) = 0,$
	$\dot{x}_2 = -x_1 + x_2(1 - x_1^2) + u,$	$x_2(0) = 1,$
	$-1 \le u(t) \le 1.$	

Hamiltonian

$$H(x,\lambda,u) = \frac{1}{2}(x_1^2 + x_2^2) + \lambda_1 x_2 + \lambda_2\big(-x_1 + x_2(1 - x_1^2) + u\big)$$

Adjoint ODEs :

$$\dot{\lambda}_1 = -H_{x_1} = -x_1 + \lambda_2(1 + 2x_1x_2), \qquad \lambda_1(t_f) = 0,$$

$$\dot{\lambda}_2 = -H_{x_2} = -x_2 - \lambda_1 - \lambda_2(1 - x_1^2), \quad \lambda_2(t_f) = 0.$$

Switching function: $\sigma = H_u = \lambda_2$

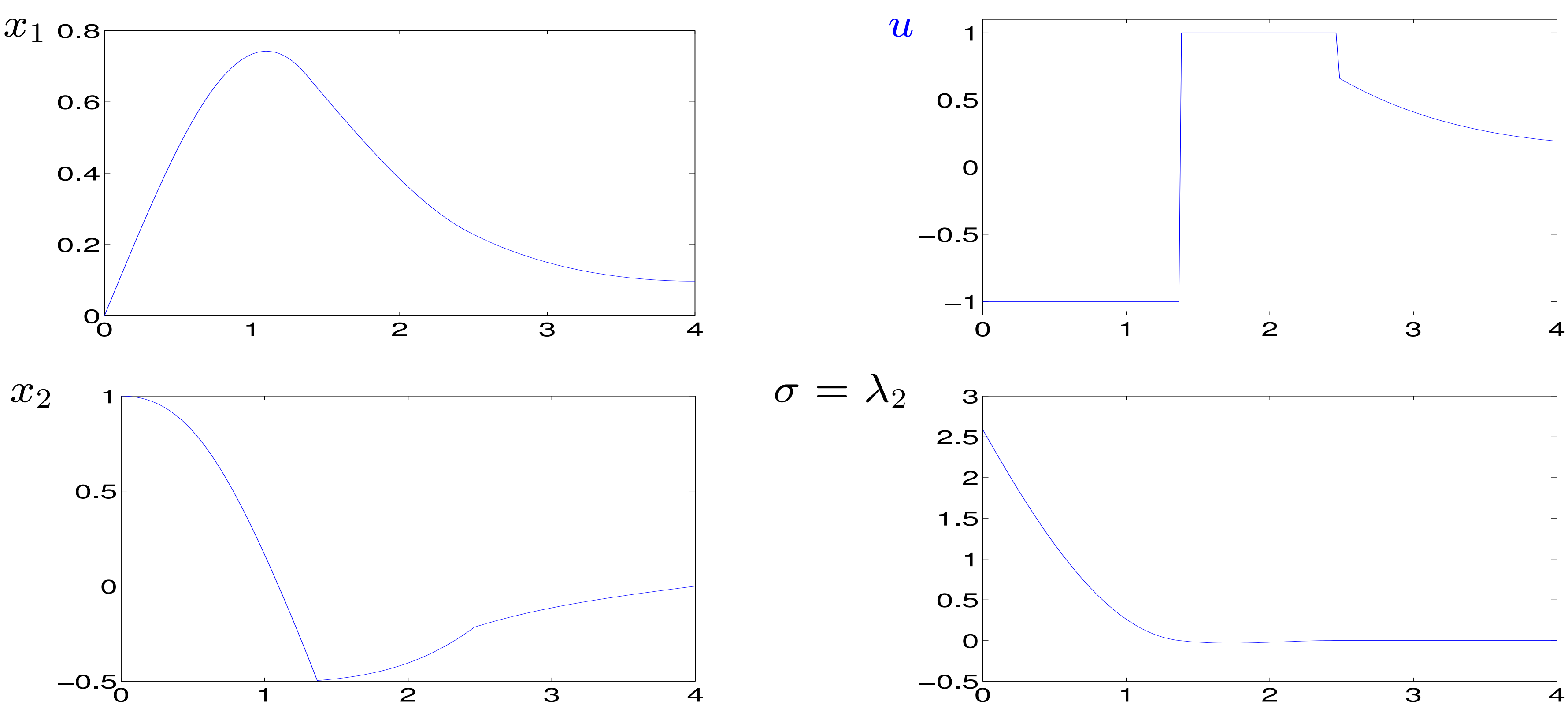
Singular feedback control of order 1 : $\sigma = \dot{\sigma} = \ddot{\sigma} \equiv 0$

$$\Rightarrow \quad u = u_{\text{sing}}(x) = 2x_1 - x_2(1 - x_1^2)$$

Van der Pol oscillator: singular control

Optimal control is bang–bang–singular

$$u = \left\{ \begin{array}{ccc} -1 & \text{for} & 0 \le t \le t_1 = 1.3667 \\ 1 & \text{for} & t_1 \le t \le t_2 = 2.4601 \\ 2x_1 - x_2(1 - x_1^2) & \text{for} & t_2 \le t \le t_f = 4 \end{array} \right\}$$



Formal SSC and sensitivity analysis

Optimize with respect to $z = (t_1, t_2)$

Hessian of the Lagrangian

$$\mathcal{L}_{zz} = \begin{pmatrix} 215.4 & -10.54 \\ -10.54 & 0.5665 \end{pmatrix} \quad \text{is positive definite.}$$

Sensitivity analysis: perturbation p in the dynamics

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + x_2(p - x_1^2) + u$$

Nominal parameter $p_0 = 1$:

$$\frac{dz}{dp}(p_0) = -(\mathcal{L}_{zz})^{-1}\mathcal{L}_{zp} = \begin{pmatrix} 0.2831 \\ 2.2555 \end{pmatrix}.$$

Time–optimal control of a semiconductor laser

Minimizing the transition time for a semiconductor laser with homogeneous transverse profile, Dokhane, Lippi, IEE Proc.-Optoelectron. **149**, 1 (2002)

$S(t)$: photon density; $N(t)$: carrier density; $I(t)$: current (control)

$$\dot{S} = \frac{dS}{dt} = -\frac{S}{\tau_p} + \Gamma G(N,S)S + \beta BN(N+P_0)$$

$$\dot{N} = \frac{dN}{dt} = \frac{I(t)}{q} - R(N) - \Gamma G(N,S)S$$

$$G(N,S) = G_p(N - N_{tr})(1 - \epsilon S) \qquad \text{(optical gain)}$$

$$R(N) = AN + BN(N+P_0) + CN(N+P_0)^2 \qquad \text{(recombination)}$$

Initial and terminal conditions (stationary points):

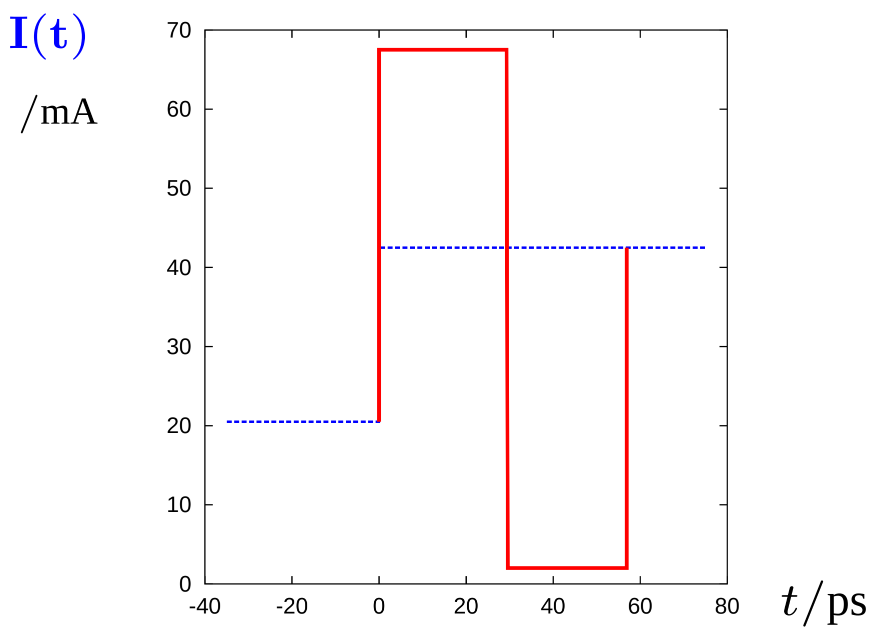
$$\begin{array}{lllllll} S(0) & = & S_0\,, & N(0) & = & N_0 & (\text{for } I(t) \equiv 20.5 \text{ mA}) \\ S(t_f) & = & S_f\,, & N(t_f) & = & N_f & (\text{for } I(t) \equiv 42.5 \text{ mA}) \end{array}$$

Semiconductor laser: time-optimal bang-bang control

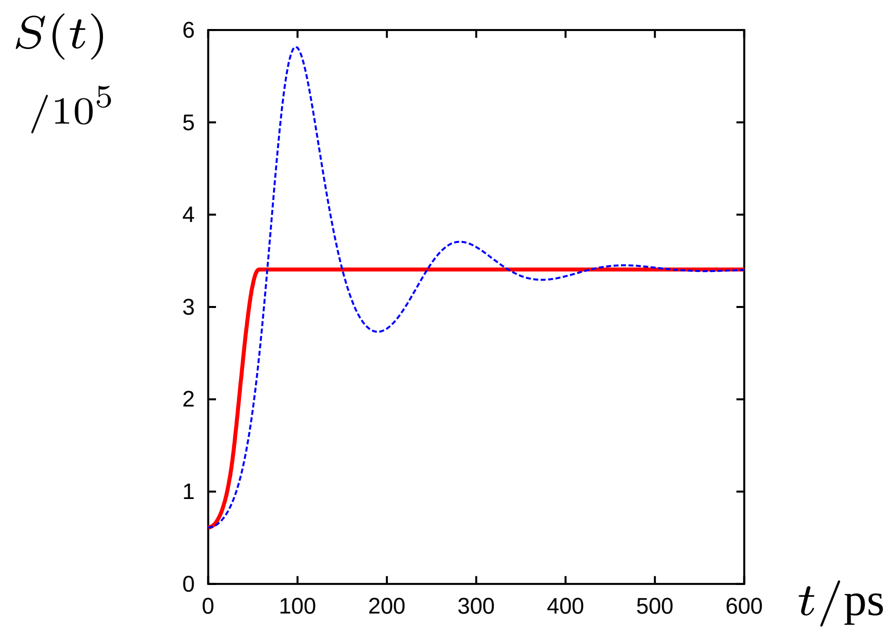
Minimize the final time t_f subject to the control bounds

$$I_{\min} \leq I(t) \leq I_{\max} \quad \text{for} \quad 0 \leq t \leq t_f\,.$$

Pertubation parameter $I_{\min}$, $I_{\max}$



time–optimal bang-bang control



Kim, Lippi, Maurer (2003)

Time–optimal bang-bang control

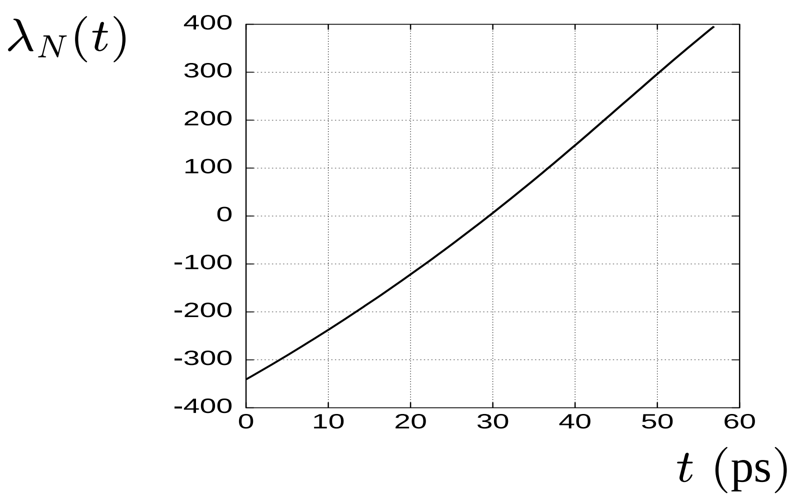
Kim, Lippi, Maure: Physica D, 191, pp. 238-260 (2004)

The time–optimal control is bang-bang

$$I(t) = \left\{ \begin{array}{lll} I_{\max} & , & 0 \le t < t_1 \,, \\ I_{\min} & , & t_1 \le t \le t_f \,. \end{array} \right\}, \quad t_1 = 29.523\,\text{ps}\,, \quad t_f = 56.894\,\text{ps}$$

Switching function and strict bang–bang property:

$$\sigma(t) = \lambda_N(t)\,, \qquad \lambda_N(t_1) = 0, \qquad \frac{d\lambda_N}{dt}(t_1) \neq 0$$



Jacobian of terminal conditions is regular:

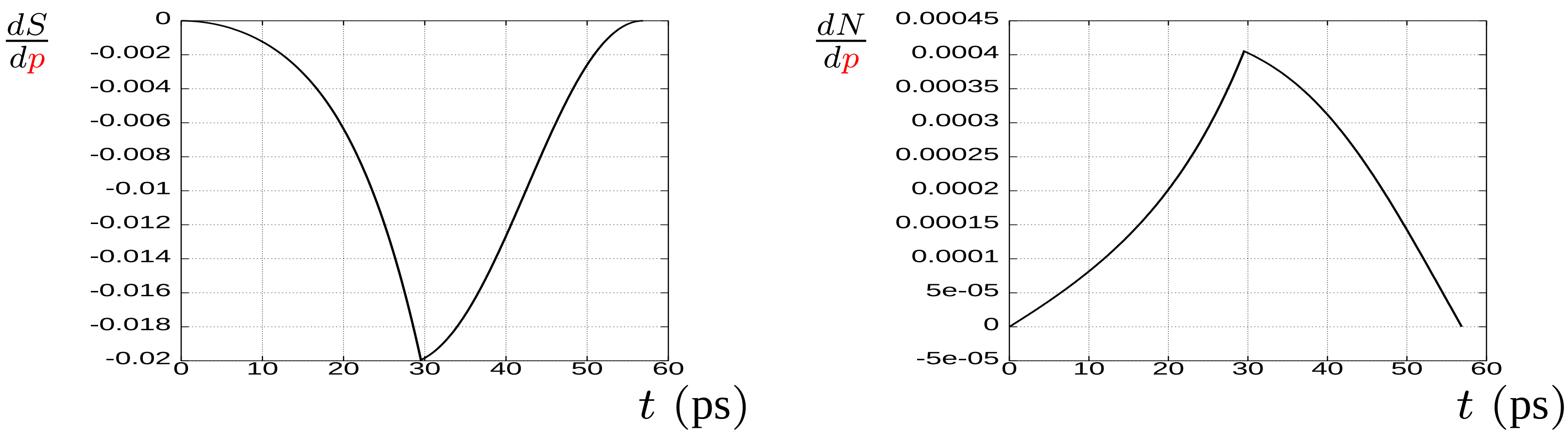
$$\Phi_z = \begin{pmatrix} 0.199855 & 0 \\ -1.5556 \cdot 10^{-4} & -2.52779 \cdot 10^{-3} \end{pmatrix}$$

First order sufficient conditions hold !

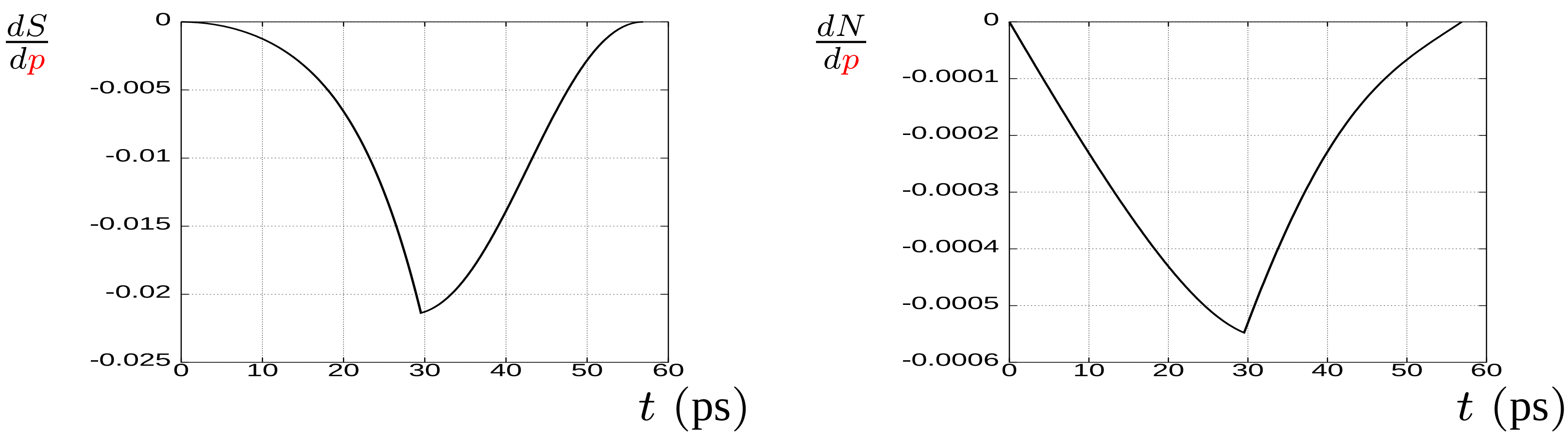
Sensitivity derivatives

$$\begin{aligned} p = I_{\max} \quad &: \quad \frac{dt_1}{dp} = -0.55486\,, \quad \frac{dt_2}{dp} = \quad 0.22419\,, \\ p = I_{\min} \quad &: \quad \frac{dt_1}{dp} = -0.24017\,, \quad \frac{dt_2}{dp} = \quad 0.57532\,. \end{aligned}$$

Sensitivity derivatives



Parametric sensitivity derivatives of the state variables (scaled) for $p = I_{\text{max}}$



Parametric sensitivity derivatives of the state variables (scaled) for $p = I_{\text{min}}$

Optimal control problem: control appearing linearly

$x(t) \in \mathbb{R}^n$	:	state variable, $0 \le t \le t_f$,
$u(t) \in \mathbb{R}$	:	control variable (scalar), piecewise continuous,
$t_f > 0$	:	final time, fixed or free,
$p \in \mathbb{R}^q$	:	perturbation parameter.

Control problem for fixed $p \in P_0 \subset \mathbb{R}^q$:

minimize $\quad g(x(0), x(t_f), t_f, p)$

subject to $\quad \dot{x}(t) = f(x(t), p) + F(x(t), p)u(t), \qquad t \in [0, t_f]\,,$

$$\varphi(x(0), x(t_f), t_f, p) = 0\,,$$

$$u_{\min} \le u(t) \le u_{\max} \qquad \forall \quad t \in [0, t_f]\,.$$

Hamiltonian function: adjoint variable $\lambda \in \mathbb{R}^n$

$$H(x, \lambda, u, p) := \lambda^* f(x, p) + \lambda^* F(x, p)u$$

Adjoint equations $\quad \dot{\lambda} = -H_x(x, \lambda, u, p) \quad$ (+ boundary conditions)

Bang–bang control

Optimal control: $u(t,p) = \text{arg min}\,\{\,H(x(t),\lambda(t),u,p)\mid u_{\min} \le u \le u_{\max}\}$

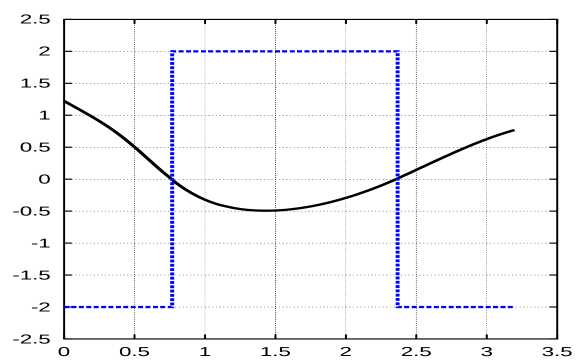
Switching function $\sigma(t,p) := H_u[t] = \lambda(t)^* F(x(t),p)$

Optimal control

$$u(t) = \begin{cases} u_{\min} & , \quad \text{if} \quad \sigma(t,p) > 0\,, \\ u_{\max} & , \quad \text{if} \quad \sigma(t,p) < 0\,, \\ \text{singular} & , \quad \text{if} \quad \sigma(t,p) \equiv 0 \quad \text{in } I_s \subset [0,t_f]\,. \end{cases}$$

Assumption: the optimal control $u_0(t)$ for the nominal parameter $p_0 \in P_0$ is bang–bang, the switching function $\sigma_0(t) = \sigma(t,p_0)$ has finitely many zeroes $0 < t_1 < t_2 < \ldots < t_s < t_f$ (switching points),

$\sigma_0(t)$



$u_0(t)$

Variables : $z := (x_0, t_1, ..., t_s, t_f) \in \mathbb{R}^n \times \mathbb{R}^{s+1},\ x(0) = x_0$

Trajectory : $x(t; x_0, t_1, ..., t_s, p)$

Parametric optimization problem:

$$OP(p) \quad \begin{cases} \text{Minimize} & G(z,p) := g(x_0, x(t_f; x_0, t_1, ..., t_s, p), t_f, p) \\ \text{subject to} & \Phi(z,p) := \varphi(x_0, x(t_f; x_0, t_1, ..., t_s, p), t_f, p) = 0 \end{cases}$$

Lagrange function in normal form:

$$\mathcal{L}(z, \rho, p) := G(z, p) + \rho^* \Phi(z, p), \quad \rho \in \mathbb{R}^{n_\varphi}$$

Nominal solution for $p = p_0$: $(z_0, \rho_0) \in \mathbb{R}^{n+s+1} \times \mathbb{R}^{n_\varphi}$

Sufficient conditions for bang–bang control, extremal field approach

- Sussmann (1987, 2001), Schättler (1988), Noble, Schättler (2001)

Second order sufficient conditions (SSC) for bang–bang control

- Sarychev (1997) : time–optimal problems
- Agrachev, Stefani, Zezza (2002) : fixed final time
- Poggiolini, Stefani (2002) : time–optimal problems
- Felgenhauer (2003), Kostyukova, Kostina (2003) : linear systems
- Osmolovski (1988, 1995, 2003), Maurer, Osmolovski (2002–2006):
  general case, equivalence of quadratic forms, representation of Lagrangian

Second order sufficient conditions (SSC)

Theorem: Suppose that the following conditions are satisfied:

- SSC hold for the nominal optimization problem $OP(p_0)$, i.e.,

  (1) $\text{rank}(\Phi_z(z_0, p_0)) = n_\varphi$,

  (2) $\mathcal{L}_z(z_0, \rho_0, p_0) = 0$,

  (3) $v^T \mathcal{L}_{zz}(z_0, \rho_0, p_0) v > 0 \quad \forall\, v \neq 0,\ \Phi_z(z_0, p_0) v = 0$.

- $\dot{\sigma}_0(t_k)(u(t_k-) - u(t_k+)) > 0, \quad k = 1, ..., s$ (strict bang–bang property).

Then the bang–bang control with s switching points $t_1, ..., t_s$ provides a strict strong minimum for the nominal optimal control problem $OC(p_0)$.

Sensitivity analysis for optimization problems: Fiacco (1976, 1983) et al.

- SSC imply stability of optimal solutions w.r.t. perturbations.

Sensitivity Theorem for Bang-Bang Control Problems

Assume that SSC are satisfied for the nominal solution (z_0, ρ_0) to the optimization problem $OC(p_0)$. Then the perturbed control problem $OC(p)$ has an optimal solution $(z(p), \rho(p))$ for all parameters p in a neighborhood of p_0 such that

(a) $(z(p_0), \rho(p_0)) = (z_0, \rho_0)$,

(b) $(z(p), \rho(p))$ is of class C^1 w.r.t. to p .

The sensitivity derivatives are given by

$$\begin{pmatrix} dz/dp \\ d\rho/dp \end{pmatrix} = - \begin{pmatrix} \mathcal{L}_{zz}(z(p), \rho(p), p) & \Phi_z(z(p), p)^* \\ \Phi_z(z(p), p) & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{L}_{zp}(z(p), \rho(p), p) \\ \Phi_p(z(p), p) \end{pmatrix}$$

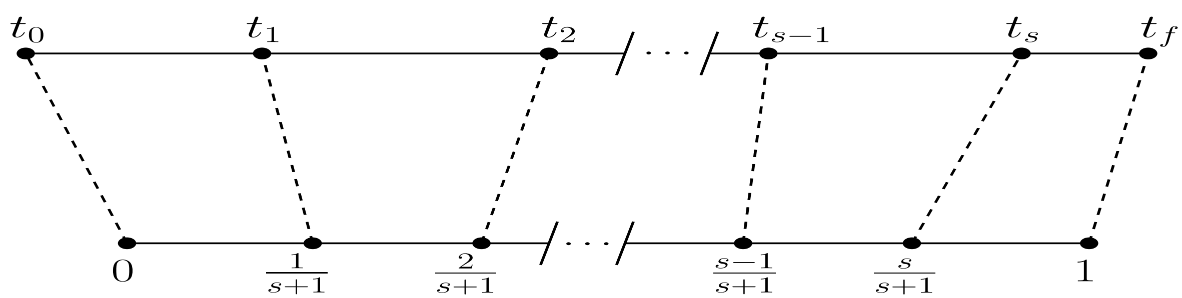
Numerical realization: code NUDOCCCS (Christof Büskens, Bremen), scaling technique of Kaya (Adalaide) et al.

Numerical methods: arc–parametrization

New optimization variables: arc lengths

$$\xi_k = t_k - t_{k-1}, \quad k = 1, .., s, s+1, \quad t_0 = 0,\ t_{s+1} = t_f$$

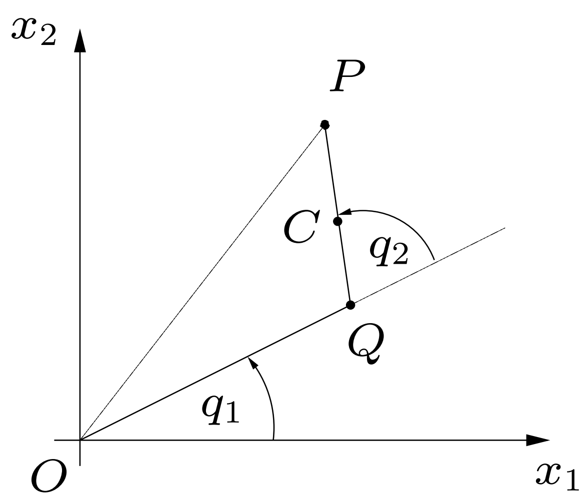
Linear mapping: $\quad t \in [t_{k-1}, t_k] \ \Leftrightarrow\ \tau \in [\, \frac{k-1}{s+1}, \frac{k}{s+1}\,]$



Transformed ODE: control: $u = u_k$ or $u = u_k(x)$ for $t_{k-1} \le t \le t_k$

$$\frac{dx}{d\tau} = (s+1)\xi_k\,[f(x(\tau), p) + F(x(\tau), p)u_k(x)\,],$$

$$\tau \in \left[\frac{k-1}{s+1}, \frac{k}{s+1}\right], \quad k = 0, 1, .., s, s+1\,.$$



Two–link robot

ODE system

$$
\begin{aligned}
\dot{q}_1 &= \omega_1 \\
\dot{q}_2 &= \omega_2 - \omega_1 \\
\dot{\omega}_1 &= \frac{1}{\Delta}(AI_{22} - BI_{12}\cos q_2) \\
\dot{\omega}_2 &= \frac{1}{\Delta}(BI_{11} - AI_{12}\cos q_2)
\end{aligned}
$$

Abbreviations

$$
\begin{aligned}
I_{11} &= I_1 + (m_2 + M)L_1^2 \\
I_{12} &= m_2 L L_1 + M L_1 L_2 \\
I_{22} &= I_2 + I_3 + M L_2^2 \\
A &= I_{12}\omega_2^2 \sin q_2 + u_1 - u_2 \\
B &= -I_{12}\omega_1^2 \sin q_2 + u_2 \\
\Delta &= I_{11}I_{22} - I_{12}^2 \cos^2 q_2
\end{aligned}
$$

Time–optimal control of a two–link robot

Boundary conditions

$$
\begin{aligned}
q_1(0) &= 0\,, & \sqrt{(x_1(t_f)-x_1(0))^2+(x_2(t_f)-x_2(0))^2} &= r\,,\\
q_2(0) &= 0\,, & q_2(t_f) &= 0\,,\\
\omega_1(0) &= 0\,, & \omega_1(t_f) &= 0\,,\\
\omega_2(0) &= 0\,, & \omega_2(t_f) &= 0\,,
\end{aligned}
$$

where $(x_1(t), x_2(t))$ are the Cartesian coordinates of the point P:

$$
\begin{aligned}
x_1(t) &= L_1 \cos q_1(t) + L_2 \cos(q_1(t)+q_2(t))\,,\\
x_2(t) &= L_1 \sin q_1(t) + L_2 \sin(q_1(t)+q_2(t))\,.
\end{aligned}
$$

Control bounds: $|u_1(t)| \le 1\,, \qquad |u_2(t)| \le 1\,, \qquad t \in [0, t_f]\,.$

Minimize the final time t_f : Hamilton function

$$
\begin{aligned}
H &= \lambda_1\omega_1 + \lambda_2(\omega_2-\omega_1) + \frac{\lambda_3}{\Delta}\left(A(u_1,u_2)I_{22} - B(u_2)I_{12}\cos q_2\right)\\
&\quad + \frac{\lambda_4}{\Delta}\left(B(u_2)I_{11} - A(u_1,u_2)I_{12}\cos q_2\right)\,.
\end{aligned}
$$

Switching functions

$$\sigma_1(t) \quad = H_{u_1}(t) = \quad \frac{1}{\Delta}(\lambda_3 I_{22} - \lambda_4 I_{12}\cos q_2)$$

$$\sigma_2(t) \quad = H_{u_2}(t) = \quad \frac{1}{\Delta}(\lambda_3(-I_{22} - I_{12}\cos q_2) + \lambda_4(I_{11} + I_{12}\cos q_2))$$

Optimal bang–bang control

$$u(t) = (u_1(t), u_2(t)) = \left\{ \begin{array}{rcl} (-1,1) & \text{for} & 0 \le t \le t_1 \\ (-1,-1) & \text{for} & t_1 \le t \le t_2 \\ (1,-1) & \text{for} & t_2 \le t \le t_3 \\ (1,1) & \text{for} & t_3 \le t \le t_4 \\ (-1,1) & \text{for} & t_4 \le t \le t_f \end{array} \right\}$$

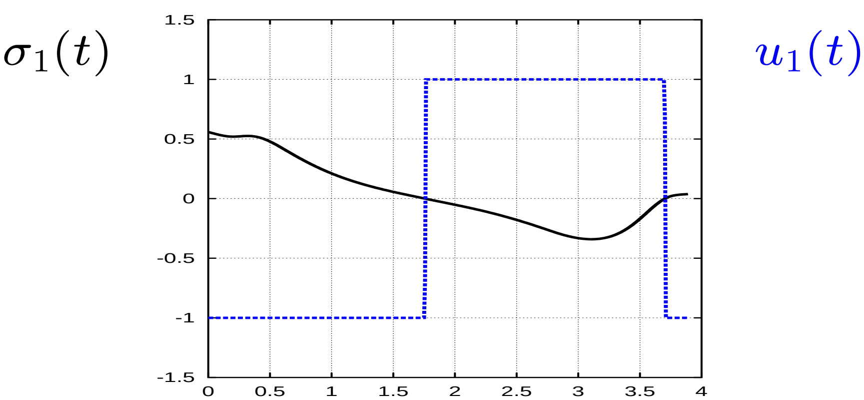
Optimal solution

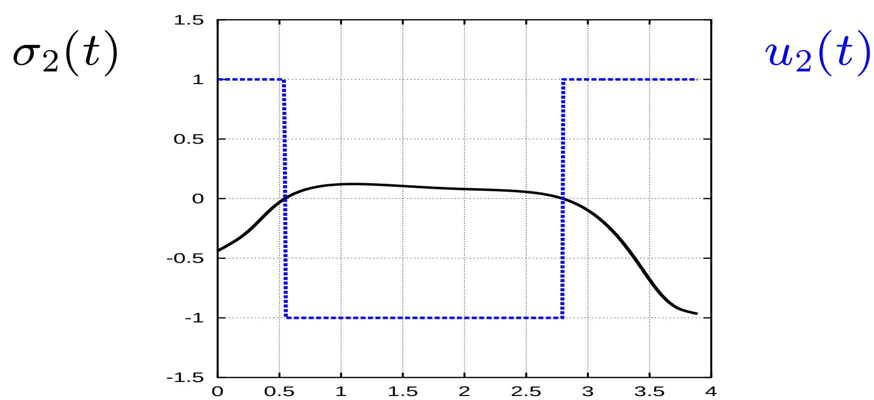
Numerical values

$$L_1 = L_2 = 1\,, \quad L = 0.5\,, \quad m_1 = m_2 = M = 1\,, \quad I_1 = I_2 = \frac{1}{3}\,, \quad r = 3$$

Switching times and final time (code NUDOCCCS, Ch. Büskens)

$$t_1 = 0.5461742\,, \quad t_2 = 1.7596815\,, \quad t_3 = 2.7983470\,,$$
$$t_4 = 3.7043862\,, \quad t_f = 3.8894093\,.$$





Second order sufficient conditions

Jacobian for terminal conditions : 4×5 matrix

$$\Phi_z(z) = \begin{pmatrix} -10.8575 & -12.7462 & -5.88332 & -1.14995 & 0 \\ 0.199280 & -2.71051 & -1.45055 & -1.91476 & -4.83871 \\ -0.622556 & 3.31422 & 2.31545 & 2.94349 & 6.19355 \\ 9.36085 & 3.03934 & 0.484459 & 0.0405811 & 0 \end{pmatrix}$$

Hessian of the Lagrangian : 5×5 matrix

$$\mathcal{L}_{zz}(z,\rho) = \begin{pmatrix} 71.1424 & 90.7613 & 42.1301 & 8.49889 & -0.0518216 \\ 90.7613 & 112.544 & 51.3129 & 10.7691 & 0.149854 \\ 42.1301 & 51.3129 & 23.9633 & 5.12403 & 0.138604 \\ 8.49889 & 10.7691 & 5.12403 & 1.49988 & 0.170781 \\ -0.0518216 & 0.149854 & 0.138604 & 0.170781 & 0.297359 \end{pmatrix}$$

Projected Hessian $\quad N^* \mathcal{L}_{zz}(z,\rho) N \approx 0.326929 > 0$

Sensitivity analysis and real–time control

Sensitivity parameter: $p = M$ (load)

Sensitivity derivatives for $p_0 = 1$

$$\frac{dt_1}{dp} = 0.0817022\,, \quad \frac{dt_2}{dp} = 0.3060921\,, \quad \frac{dt_3}{dp} = 0.7115999\,,$$
$$\frac{dt_4}{dp} = 0.7310003\,, \quad \frac{dt_f}{dp} = 0.8498167\,.$$

Real–time control

$$u(t) = u_k \quad \text{or} \quad u(t) = u_k(x(t))\,, \qquad t_{k-1}(p) \le t \le t_k(p)$$

Real–time approximation

$$t_k(p) \approx t_k(p_0) + \frac{dt_k}{dp}(p_0)(p - p_0)$$

D. I. Cho, P.L. Abad and M. Parlar,
Optimal Production and Maintenance Decisions when a System Experiences Age–Dependent Deterioration, Optimal Control Appl. Meth. **14**, 153–167 (1993)

State and control variables:

$x(t)$	:	inventory level at time $t \in [0, t_f]$, final time t_f is fixed,
$y(t)$	:	proportion of 'good' units of end items produced: process performance,
$u(t)$	:	scheduled production rate (control),
$m(t)$	:	preventive maintenance rate to reduce the proportion of defective units produced (control),
$\alpha(t)$	:	obsolescence rate of the process performance in the absence of maintenance, non–decreasing in time,
$s(t)$	:	demand rate,
$\rho = 0.1$	:	discount rate,

State equations:

$$\dot{x}(t) = y(t)u(t) - s(t), \qquad x(0) = x_0, \quad x(t_f) = 0,$$
$$\dot{y}(t) = -\alpha(t)y(t) + (1 - y(t))m(t), \quad y(0) = y_0.$$

Control constraints : $0 \le u(t) \le U\,, \quad 0 \le m(t) \le M\,, \quad 0 \le t \le t_f$

State constraint : $h(y(t)) := y(t) - y_{\min} \ge 0$

Data : $s(t) = 4\,,\ \alpha(t) = 2\,,\ x_0 = 3,\ y_0 = 1\,,\ U = 3\,,\ M = 4$

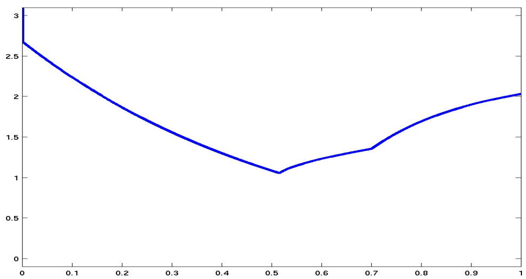
Maximize $$F(x, y, u, m) = 10\, y(t_f)e^{-\rho t_f} + $$
$$+ \int_0^{t_f} e^{-\rho t}\,[\,8s(t) - x(t) - (ru^2(t) + qu(t)) - 2.5m(t)\,]\,dt\,.$$

L^2–functional in u for $r = 2\,,\ q = 0$: mixed type of control (Osmolovskii/M.)
L^1–functional in u for $r = 0\,,\ q = 4$: this talk

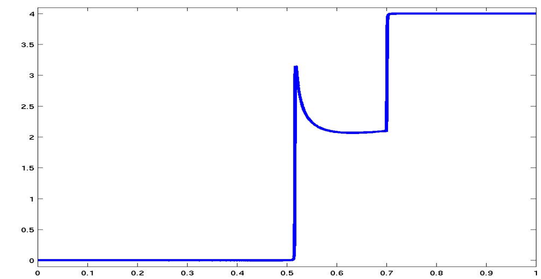
L^2–Functional: bang–singular maintenance

L^2 functional in u : initial values $x_0 = 3$, $y_0 = 1$, final time $t_f = 1$

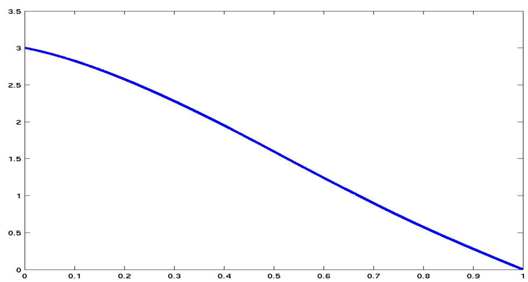
$u(t)$



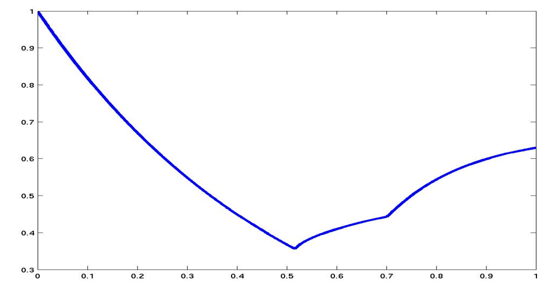
$m(t)$



$x(t)$



$y(t)$



Sufficient conditions are not available !

L^1–Functional: necessary conditions

Current value Hamiltonian for maximum principle: adjoint variables λ_x, λ_y

$$\begin{aligned} H(x, y, u, m, \lambda_x, \lambda_y) = &\ (8s - x - 4u - 2.5m) \\ &+ \lambda_x(yu - s) + \lambda_y(-\alpha y + (1-y)m)), \end{aligned}$$

Adjoint equations and transversality conditions:

$$\begin{aligned} \dot{\lambda}_x &= \rho\lambda_x - \tfrac{\partial H}{\partial x} = \rho\lambda_x + h, & \lambda_x(t_f) &= \nu, \\ \dot{\lambda}_y &= \rho\lambda_y - \tfrac{\partial H}{\partial y} = \lambda_y(\rho + \alpha + m) - \lambda_x u, & \lambda_y(t_f) &= 10\,. \end{aligned}$$

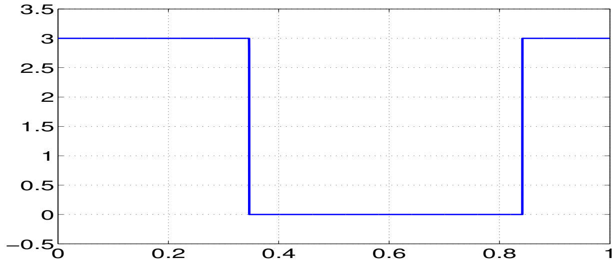
Switching functions $\sigma^u(t) = \lambda_x(t)y(t) - 4$, $\sigma^m(t) = \lambda_y(t)(1 - y(t)) - 2.5$

$$u(t) = \left\{ \begin{array}{lll} 0 & , \ \text{if} & \sigma^u(t) < 0, \\ U = 3 & , \ \text{if} & \sigma^u(t) > 0 \\ \text{singular} & , \ \text{if} & \sigma^u(t) \equiv 0 \end{array} \right\}, \quad m(t) = \left\{ \begin{array}{lll} 0 & , \ \text{if} & \sigma^m(t) < 0, \\ M = 4 & , \ \text{if} & \sigma^m(t) > 0 \\ \text{singular} & , \ \text{if} & \sigma^m(t) \equiv 0 \end{array} \right\}.$$

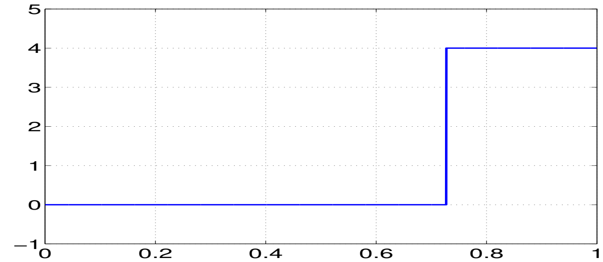
$t_f = 1$: bang–bang controls u and m

Solution for $t_f = 1$: controls u and m are bang–bang

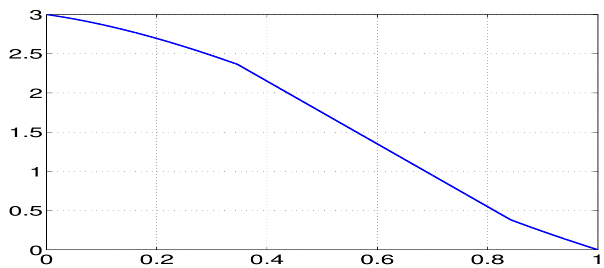
$u(t)$



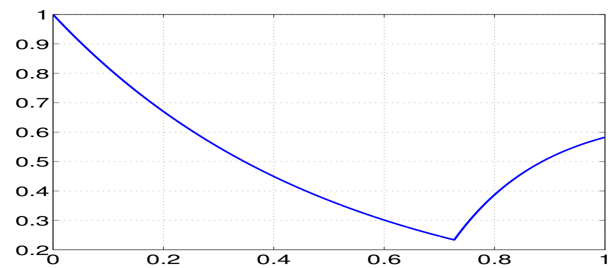
$m(t)$



$x(t)$



$y(t)$

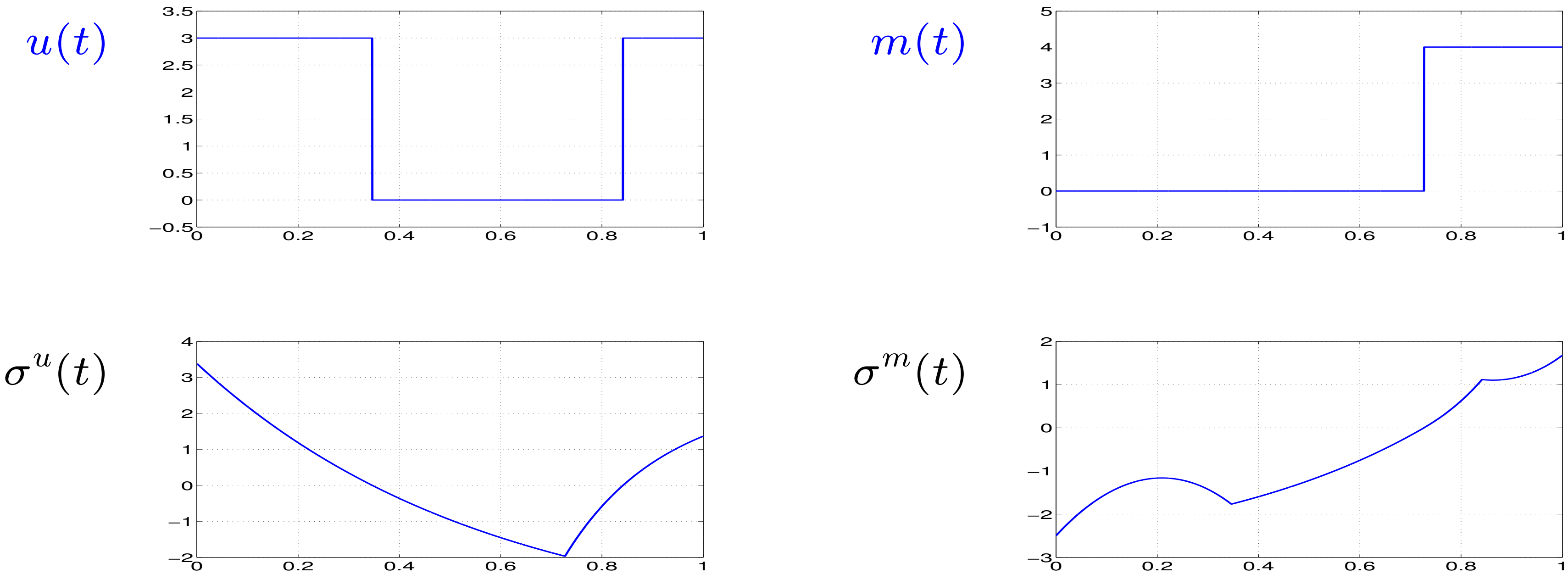


Switching times: $t_1 = 0.3465,\ t_2 = 0.7270,\ t_3 = 0.8415$

Optimization problem: optimize $z := (t_1, t_2, t_3)$, boundary condition $x(t_f, z) = 0$.

SSC hold : $D^2_{zz}\mathcal{L}$ is positive definite on the tangent space of the constraint.

$t_f = 1$: SSC for bang–bang controls u and m



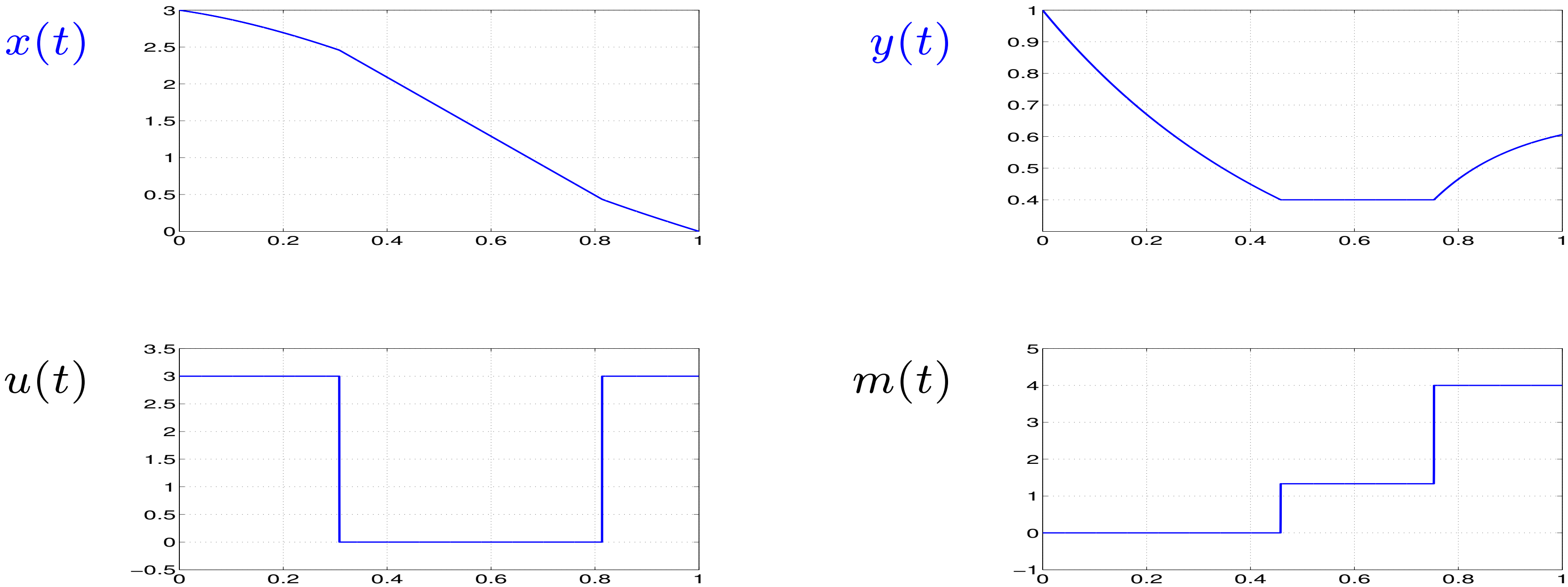
Strict bang-bang property holds: $\dot{\sigma}^u(t_1) < 0\,, \;\; \dot{\sigma}^m(t_2) > 0\,, \;\; \dot{\sigma}^u(t_3) > 0\,.$

$\Rightarrow$ SSC hold for the control problem:

Agrachev, Stefani, Zezza (SICON 2002),

Osmolovskii (1988), Osmolovskii, Maurer (2003–05)

state constraint $y(t) \geq 0.4$



Switching times: $t_1 = 0.3080\,,\ t_2 = 0.4581\,,\ t_3 = 0.7531\ \ t_4 = 0.8137$

Boundary arc $[t_2, t_3]$: boundary control $m_b \equiv \alpha\, y_{\min}/(1 - y_{\min})$ (feedback)

Optimization problem: optimize $z := (t_1, t_2, t_3, t_4)$

Boundary and entry conditions: $x(t_f, z) = 0,\ y(t_2, z) = 0.4$

SSC hold : $D^2_{zz}\mathcal{L}$ is positive definite on the tangent space of the constraints.

References

H. MAURER, J.-H.R. KIM, AND G.VOSSEN: On a state constrained constrained control problem in optimal production and maintenance, in: Optimal Control and Dynamic Games, Applications in Finance, Management Science and Economics (C. Deissenberg, R.F. Hartl, eds), pp. 289–308, Springer, 2005

H. MAURER, C. BÜSKENS, J.-H.R. KIM, AND Y.C. KAYA: Optimization methods for the numerical verification of second order sufficient conditions for bang-bang controls, *Optimal Control Application and Methods* **26**, pp. 129–156 (2005).

GODDARD problem: high altitude rocket

A.E. Bryson, Y.C. Ho: *Applied Optimal Control*, Ginn and Company, 1969.

H. Maurer: *Numerical solution of singular control problems using multiple shooting techniques*, J. of Optimization Theory **18**, pp. 235–257 (1976).

Maximize $h(t_f)$ (final time t_f is free)

subject to
$$\dot h = v,$$
$$\dot v = \frac{1}{m}\Big(cu - D(v,h)\Big) - \gamma(h),$$
$$\dot m = -u,$$
$$h(0) = v(0) = 0, \quad m(0) = m_0, \quad m(t_f) = m_f,$$
$$0 \le u(t) \le u_{\max} = 9.52551.$$

Drag $D(v,h) = \alpha v^2 \exp(-\beta h)$, Gravitation $\gamma(h) = g_0 \frac{r_0^2}{(r_0+h)^2}$.

Data : $\alpha = 0.01227,\ \beta = 0.145 \times 10^{-3},\ c = 2060,$

$m_0 = 214.839,\ m_f = 67.9833,\ g_0 = 9.81,\ r_0 = 6.371 \times 10^6\,[m]$

GODDARD problem: singular feedback control

Hamiltonian:

$$H(h, v, m, \lambda, u) = \lambda_h v + \lambda_v \Big(\frac{1}{m} (c \cdot u - D(v, h)) - \gamma(h) \Big) - \lambda_m u.$$

Switching function:

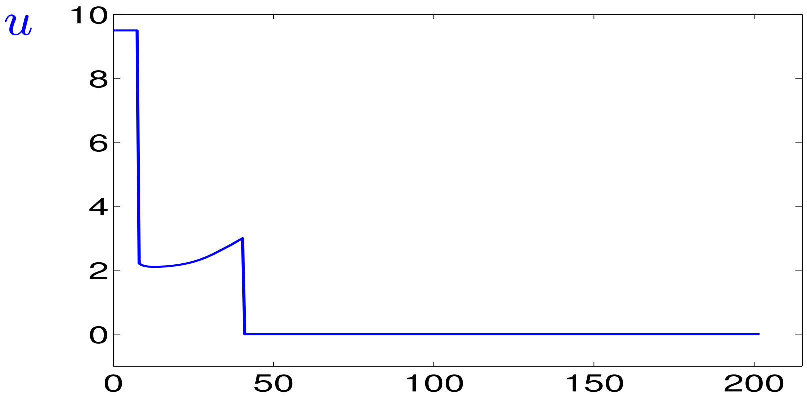
$$\sigma = H_u = c \frac{\lambda_v}{m} - \lambda_m .$$

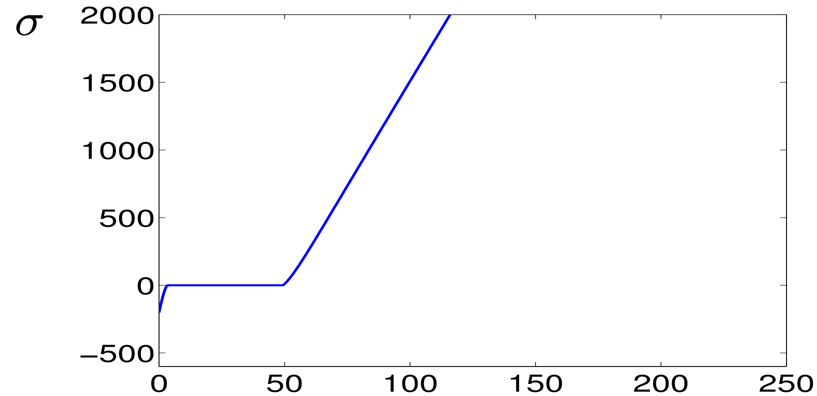
Singular feedback control of order 1: $\sigma \equiv \dot{\sigma} \equiv \ddot{\sigma} \equiv 0$

$$u_{\text{sing}}(h, v, m) = \frac{D}{c} + \frac{m(c - v)D_h + D_v \gamma + c D_{vv} \gamma - c D_{vh} v + c m \gamma_h}{D + 2cD_v + c^2 D_{vv}}$$

GODDARD problem: bang–singular–bang control

Control structure: $u(t) = u^{\max}$ | singular | 0





Switching times and final time:

$$t_1 = 4.11525, \quad t_2 = 46.04063, \quad t_f = 212.90299.$$

GODDARD problem: SSC w.r.t. t_1, t_2, t_f

Optimization : arc lengths $\xi_1 = t_1,\ \xi_2 = t_2 - t_1,\ \xi_3 = t_f - t_2$

$$\tilde{\mathcal{L}}_{\xi\xi} = \begin{pmatrix} -244422.57 & -19472.17 & -1611.55 \\ -19472.17 & -1488.13 & -119.50 \\ -1611.55 & -119.50 & 9.36 \end{pmatrix},$$

$$\tilde{\Phi}_\xi = \begin{pmatrix} -55.76 & -4.20 & 0.00 \end{pmatrix}.$$

$\tilde{\mathcal{L}}_{\xi\xi}$ is positive definite on ker($\tilde{\Phi}_\xi$) : reduced Hessian

$$\tilde{N}^* \tilde{\mathcal{L}}_{\xi\xi} \tilde{N} = \begin{pmatrix} 58.04 & 1.77 \\ 1.77 & 9.36 \end{pmatrix}$$

where the columns of $\tilde{N}$ span ker($\tilde{\Phi}_\xi$).

- Optimal control of a chemical batch–reaction
- Optimal chemotherapy using compartment models (Ledzewicz, Schättler,M.)
- Optimal control of an underwater vehicle; 10 switching points and chattering (Chyba, Sussmann, Vossen, M.)
- Extension to state constrained problems (with Ledzewicz, Schättler, M., Vossen)
- Nonsmooth L^1–minimization: Minimize $\int_0^{t_f} |u(t)|\,dt$ (Vossen)
- Open challenge problem : Bang–singular controls (Vossen, Dmitruk, Stefani)
- Bang-bang and singular control (boundary and distributed control) for elliptic and parabolic PDEs (M., Theißen)